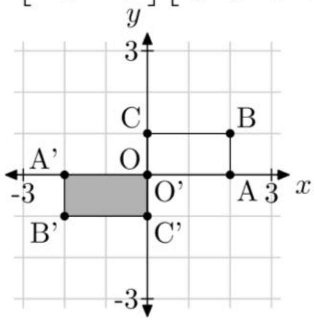
# Specialist Mathematics Unit2: **Chapter 11**

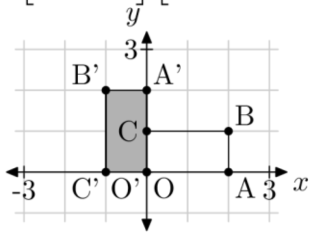
## Ex 11A

1. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$



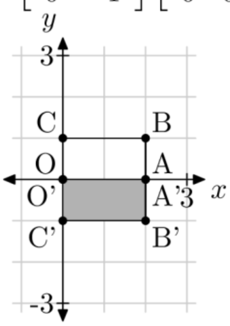

This represents a 180° rotation.

2. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$



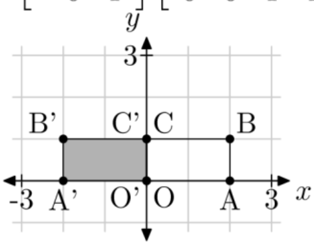

This represents a 90° anticlockwise rotation.

3. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$



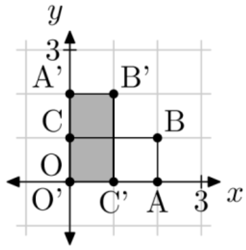

This represents a reflection in the $x-$axis.

4. $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$



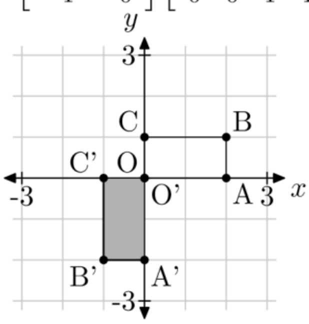

This represents a reflection in the $y-$axis.

5. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \end{bmatrix}$



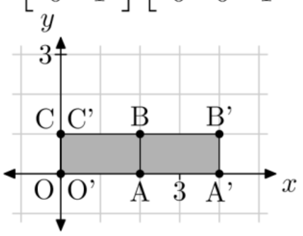

This represents a reflection in the line $y = x$.

6. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & -2 & -2 & 0 \end{bmatrix}$



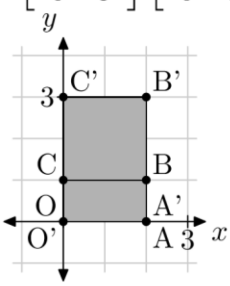

This represents a reflection in the line $y = -x$.

7. $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$



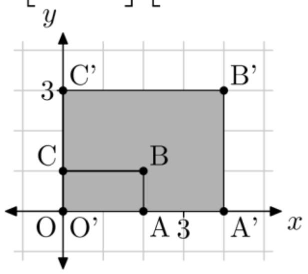

This represents a horizontal dilation of factor 2.

8. $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



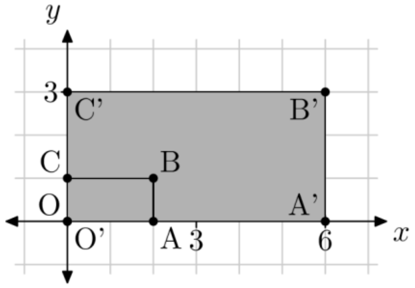

This represents a vertical dilation of factor 3.

9. $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



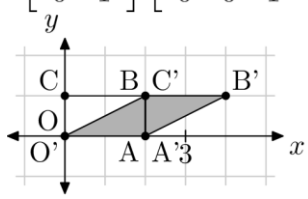

This represents a dilation with a horizontal scale factor of 2 and vertical scale factor of 3.

10. $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 6 & 0 \\ 0 & 0 & 3 & 3 \end{bmatrix}$



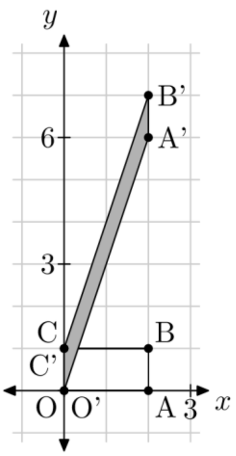

This represents a dilation with uniform scale factor of 3.

11. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$




This represents a shear parallel to the $x-$axis with scale factor of 2.

12. $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 & 0 \\ 0 & 6 & 7 & 1 \end{bmatrix}$




This represents a shear parallel to the $y-$axis with scale factor of 3.

13. The working needed here is quite straightforward. I present a worked solution for the first matrix only.

(a) $\det \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1 \times -1 + 0 \times 0 = 1$

(b)
$$\text{Area OABC} = 2 \times 1 = 2$$
$$\text{Area O}'\text{A}'\text{B}'\text{C}' = 2 \times 1 = 2$$
$$\frac{\text{Area O}'\text{A}'\text{B}'\text{C}'}{\text{Area OABC}} = \frac{2}{2} = 1$$

# Ex 11B

1. (a) For matrix A, $(1,0)$ maps to $(0,-1)$ and $(0,1)$ maps to $(1,0)$; the required matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

For matrix B, $(1,0)$ maps to $(-1,0)$ and $(0,1)$ maps to $(0,-1)$; the required matrix is

$$B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

For matrix C, $(1,0)$ maps to $(0,1)$ and $(0,1)$ maps to $(-1,0)$; the required matrix is

$$C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(b)
$$\begin{aligned} A^2 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0^2 + 1 \times -1 & 0 \times 1 + 1 \times 0 \\ -1 \times 0 + 0 \times -1 & -1 \times 1 + 0^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= B \end{aligned}$$

(c)
$$\begin{aligned} C^2 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0^2 - 1 \times 1 & 0 \times -1 - 1 \times 0 \\ 1 \times 0 + 0 \times 1 & 1 \times -1 + 0^2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= B \end{aligned}$$

(d)
$$\begin{aligned} A^3 &= A^2A \\ &= BA \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \times 0 + 0 \times -1 & -1 \times 1 + 0^2 \\ 0^2 + (-1)^2 & 0 \times 1 - 1 \times 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= C \end{aligned}$$

(e)
$$\begin{aligned} B^2 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^2 \\ &= \begin{bmatrix} (-1)^2 + 0^2 & -1 \times 0 + 0 \times -1 \\ 0 \times -1 - 1 \times 0 & 0^2 + (-1)^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

(f)
$$\begin{aligned} A^{-1} &= \frac{1}{0^2 - (-1 \times 1)}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= C \end{aligned}$$

(Alternatively, show that $AC = I$)

(g)
$$\begin{aligned} B^{-1} &= \frac{1}{(-1)^2 - 0^2}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\ &= B \end{aligned}$$

Alternatively, since we have already shown that $B^2 = I$,

$$\begin{aligned} B^2 &= I \\ B^{-1}B^2 &= B^{-1}I \\ (B^{-1}B)B &= B^{-1} \\ IB &= B^{-1} \\ B &= B^{-1} \end{aligned}$$

2. (a) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) A reflection in the $x-$axis followed by a reflection in the $y-$axis is represented by premultiplying the matrix for the first reflection by the matrix for the second, i.e.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

A reflection in the $y-$axis followed by a reflection in the $x-$axis is represented by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(e) Compare the results from (d) and (e).

3. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

The transformation matrix is P= $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

If P is its own inverse, then $\mathrm{P}^2 = \mathrm{I}$.

$$\begin{aligned} \mathrm{P}^2 &= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \mathrm{I} \end{aligned}$$

4. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ maps to $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ maps to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The transformation matrix is $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

The determinant of this matrix is $3 \times 1 - 0 \times 0 = 3$ as expected.

5. (a) No working needed.
   (b) No working needed. (A, B, C and D are the columns of the second matrix and A′, B′, C′ and D′ are the columns of the product.)

6.
$$\begin{aligned} \mathrm{TA} &= \mathrm{A}' \\ \mathrm{T}^{-1}\mathrm{TA} &= \mathrm{T}^{-1}\mathrm{A}' \\ \mathrm{A} &= \mathrm{T}^{-1}\mathrm{A}' \end{aligned}$$

$$\begin{aligned} \mathrm{T}^{-1} &= \frac{1}{1 \times 1 - 2 \times 0} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \\ \mathrm{A} &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ \mathrm{B} &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathrm{C} &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -3 \end{bmatrix} \end{aligned}$$

A, B and C have coordinates $(1, 3)$, $(1, 1)$ and $(4, -3)$ respectively.

7.
$$\begin{aligned} \mathrm{T}^{-1} &= \frac{1}{2 \times 1 - 0 \times -3} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \\ \mathrm{A} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 2 \\ 6 \end{bmatrix} \\ \mathrm{B} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -2 \\ 4 \end{bmatrix} \\ \mathrm{C} &= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{aligned}$$

A, B and C have coordinates $(1, 3)$, $(-1, 2)$ and $(0, 2)$ respectively.

8.
$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathrm{P} &= \mathrm{P}' \\ \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \mathrm{P}' &= \mathrm{P}'' \\ \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathrm{P} &= \mathrm{P}'' \\ \begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix} \mathrm{P} &= \mathrm{P}'' \end{aligned}$$

Matrix $\begin{bmatrix} 4 & 1 \\ 1 & 0 \end{bmatrix}$ will transform PQR directly to P″Q″R″.

9.
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}^{-1} &= \frac{1}{-1-0} \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix} \end{aligned}$$

Matrix $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transform PQR directly to P″Q″R″.

Matrix $\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$ will transform P″Q″R″ directly to PQR. (The matrix is its own inverse.)

10. A shear parallel to the $y$-axis, scale factor 3, transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$.

A clockwise rotation of 90° about the origin transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

The single matrix to perform both these transformations in sequence is

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$$

12. Post-multiply both sides of the equation with the inverse of $\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$ to eliminate it from the LHS:

$$\begin{aligned}
\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} \\
\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}^{-1} &= \frac{1}{1+6}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\
\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 12 & -1 \\ 7 & 0 \end{bmatrix}\frac{1}{7}\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \\
&= \frac{1}{7}\begin{bmatrix} 14 & 35 \\ 7 & 21 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}
\end{aligned}$$

so $a = 2$, $b = 5$, $c = 1$ and $d = 3$.

13. (a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ -1 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{1-0}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

(d) First, to transform $A_3B_3C_3D_3$ to $A_2B_2C_2D_2$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} = \frac{1}{0+1}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then to further transform the result to $A_1B_1C_1D_1$ we use the matrix we obtained in (c), so the single matrix that combines both is

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$$

14. A reflection in the $x$-axis transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

A reflection in the line $y = x$ transforms $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and so is represented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

A 90° clockwise rotation is represented by $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (see question 10).

The matrix that represents these three transformations in sequence is

$$\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

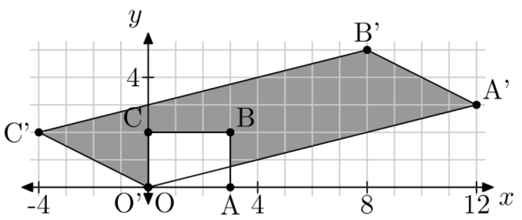
which is the identity matrix, resulting in the original shape in the original position.

15. (a) $\det \mathrm{T} = 4 \times 1 - (-2) \times 1 = 6$. Given that the area of OABC is 6 units$^2$, the area of O′A′B′C′ is $6 \times 6 = 36$ units$^2$.

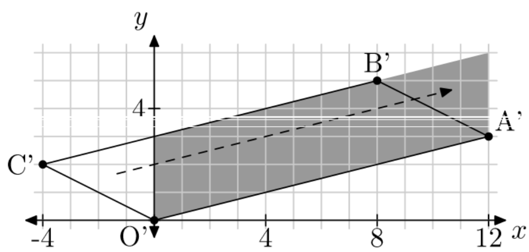
(b) $\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 3 & 3 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 12 & 8 & -4 \\ 0 & 3 & 5 & 2 \end{bmatrix}$

The coordinates of O′, A′, B′ and C′ are $(0, 0)$, $(12, 3)$, $(8, 5)$ and $(-4, 2)$ respectively.

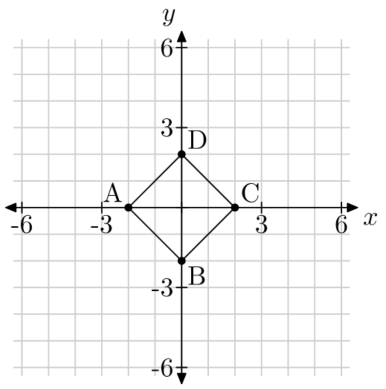
(c)




(d) There are number of straightforward ways of determining the area of the parallelogram. For example if we slice off the part of the parallelogram that is left of the $y$-axis and slide it to the other end (as shown below), we get a parallelogram with a (vertical) base of 3 and (horizontal) perpendicular height of 12, yielding an area of 36.



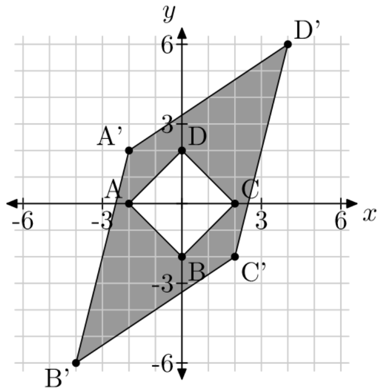
16. (a)




(b) Area = 8 units$^2$ (area of any square, rhombus or kite is half the product of its diagonals).

(c) $\det \mathrm{M} = 1 \times 3 - 2 \times -1 = 5$. The area of A′B′C′D′ is $5 \times 8 = 40$ units$^2$.

(d) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 2 & 4 \\ 2 & -6 & -2 & 6 \end{bmatrix}$




Area = 40 units$^2$.

17. Every point on the line $y = 2x + 3$ can be represented by $\begin{bmatrix} x \\ 2x+3 \end{bmatrix}$.

To prove:

$$\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ 2x+3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

for all $x$.

Proof:

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ 2x+3 \end{bmatrix} \\ &= \begin{bmatrix} 2(x) - (2x+3) \\ -2(x) + (2x+3) \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 3 \end{bmatrix} \\ &= \text{RHS} \end{aligned}$$

Notice that the matrix $\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$ is singular (i.e. it has a determinant of zero) and therefore is not invertable. This is a requirement of any matrix that transforms two or more distinct points to the same position in the image.

18. Every point on the line $y = x - 1$ can be represented by $\begin{bmatrix} x \\ x-1 \end{bmatrix}$.

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ x-1 \end{bmatrix} &= \begin{bmatrix} x \\ 2(x) + (x-1) \end{bmatrix} \\ &= \begin{bmatrix} x \\ 3x-1 \end{bmatrix} \end{aligned}$$

The equation of the image line is $y = 3x - 1$.

19. To prove:

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix}$$

for all $a$, $b$ and for some relationship between $x$ and $a$ and $b$.

Proof:

$$\begin{aligned} \text{LHS} &= \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} a+3b \\ 3a+9b \end{bmatrix} \\ &= \begin{bmatrix} a+3b \\ 3(a+3b) \end{bmatrix} \end{aligned}$$

Let $x = a + 3b$

$$\begin{aligned} \text{then LHS} &= \begin{bmatrix} x \\ 3x \end{bmatrix} \\ &= \text{RHS} \end{aligned}$$

20. (a) $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ 5-3x \end{bmatrix} = \begin{bmatrix} 6(x) + 2(5-3x) \\ 3(x) + (5-3x) \end{bmatrix}$

$$= \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The line $y = 5 - 3x$ is transformed to the point $(10, 5)$.

(b) $\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 6a+2b \\ 3a+b \end{bmatrix}$

Let $x = 6a + 2b$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ \frac{x}{2} \end{bmatrix}$$

Points on the $x$-$y$ plane are transformed to the line $y = \frac{x}{2}$ or $2y = x$.

21. Let $(a, b)$ be an arbitrary point before transformation and $(a', b')$ the corresponding point after transformation.

$$\begin{aligned} \begin{bmatrix} a' \\ b' \end{bmatrix} &= \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \\ &= \begin{bmatrix} 3a \\ 2a+b \end{bmatrix} \end{aligned}$$

If the point before transformation lies on the line $y = m_1 x + p$ then $b = m_1 a + p$ and the transformed point is

$$\begin{aligned}\begin{bmatrix} a' \\ b' \end{bmatrix} &= \begin{bmatrix} 3a \\ 2a + (m_1 a + p) \end{bmatrix} \\ &= \begin{bmatrix} 3a \\ (m_1 + 2)a + p \end{bmatrix}\end{aligned}$$

We can turn this into a pair of parametric equations then convert that to a Cartesian equation of a line:

$$\begin{aligned} x &= 3a \\ y &= (m_1 + 2)a + p \\ &= \frac{(m_1 + 2)(3a)}{3} + p \\ &= \frac{m_1 + 2}{3} x + p \end{aligned}$$

which is in the form $y = m_2 x + p$ where $m_2 = \frac{m_1+2}{3}$, as required.

Now consider two lines perpendicular to each other both before and after transformation.

Let $q$ be the gradient of the first line before transformation.

Since the lines are perpendicular, the gradient of the second line is $-\frac{1}{q}$.

Transforming the first line results in a gradient of $\frac{q+2}{3}$.

Transforming the second line results in a gradient of $\frac{-\frac{1}{q}+2}{3} = \frac{-1+2q}{3q}$.

Since the lines a perpendicular after transformation,

$$\begin{aligned} \frac{q+2}{3} &= -\frac{3q}{-1+2q} \\ &= \frac{3q}{1-2q} \\ (q+2)(1-2q) &= 9q \\ q - 2q^2 + 2 - 4q &= 9q \\ -2q^2 + 2 - 12q &= 0 \\ q^2 - 1 + 6q &= 0 \\ q^2 + 6q - 1 &= 0 \\ (q+3)^2 - 9 - 1 &= 0 \\ (q+3)^2 &= 10 \\ q + 3 &= \pm\sqrt{10} \\ q &= -3 \pm \sqrt{10} \end{aligned}$$

Hence the gradients of the two lines before transformation are $-3 + \sqrt{10}$ and $-3 - \sqrt{10}$.

# Ex 11C

1. (a) $$\begin{aligned}\begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} &= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\end{aligned}$$

(b) $$\begin{aligned}\begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} &= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \\ &= \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\end{aligned}$$

(c) $$\begin{aligned}\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\end{aligned}$$

(d) $$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

(e) Two consecutive 30° anticlockwise rotations about the origin are represented by

$$\begin{aligned} &\frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} 3-1 & -2\sqrt{3} \\ 2\sqrt{3} & -1+3 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \end{aligned}$$

which is a 60° anticlockwise rotation about the origin.

(f) A 30° anticlockwise rotation about the origin followed by a 60° anticlockwise rotation about the origin is represented by

$$\begin{aligned} &\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} \sqrt{3}-\sqrt{3} & -1-3 \\ 3+1 & -\sqrt{3}+\sqrt{3} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

which is a 90° anticlockwise rotation about the origin.

(g) Two consecutive 45° anticlockwise rotations about the origin are represented by

$$\begin{aligned} &\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{2}{4}\begin{bmatrix} 1-1 & -1-1 \\ 1+1 & -1+1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

which is a 90° anticlockwise rotation about the origin.

2. (a) $\begin{bmatrix} \cos(2\times 30) & \sin(2\times 30) \\ \sin(2\times 30) & -\cos(2\times 30) \end{bmatrix}$

$$= \begin{bmatrix} \cos 60 & \sin 60 \\ \sin 60 & -\cos 60 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$$

(b) $\begin{bmatrix} \cos(2\times 60) & \sin(2\times 60) \\ \sin(2\times 60) & -\cos(2\times 60) \end{bmatrix}$

$$= \begin{bmatrix} \cos 120 & \sin 120 \\ \sin 120 & -\cos 120 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

For (a),

$$\left(\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\right)^2 = \frac{1}{4}\begin{bmatrix} 1+3 & \sqrt{3}-\sqrt{3} \\ \sqrt{3}-\sqrt{3} & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly for (b),

$$\left(\frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\right)^2 = \frac{1}{4}\begin{bmatrix} 1+3 & -\sqrt{3}+\sqrt{3} \\ -\sqrt{3}+\sqrt{3} & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Any reflection must be its own inverse since reflecting a reflection restores the original. Consider the general form for a reflection:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}^2 = \begin{bmatrix} \cos^2 2\theta + \sin^2 2\theta & \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta \\ \cos 2\theta \sin 2\theta - \cos 2\theta \sin 2\theta & \sin^2 2\theta + \cos^2 2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. $\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

5. A rotation of angle $A$ followed by a rotation of angle $B$ is equivalent to a rotation of angle $A+B$.

A rotation of angle $A$ followed by a rotation of angle $B$ is represented by $\begin{bmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{bmatrix}\begin{bmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{bmatrix}$

$$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -\cos A\sin B - \sin A\cos B \\ \sin A\cos B + \cos A\sin B & -\sin A\sin B + \cos A\cos B \end{bmatrix}$$

$$= \begin{bmatrix} \cos A\cos B - \sin A\sin B & -(\sin A\cos B + \cos A\sin B) \\ \sin A\cos B + \cos A\sin B & \cos A\cos B - \sin A\sin B \end{bmatrix}$$

A single rotation of angle $A+B$ is represented by $\begin{bmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{bmatrix}$

Equating these gives

$$\begin{bmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{bmatrix} = \begin{bmatrix} \cos A\cos B - \sin A\sin B & -(\sin A\cos B + \cos A\sin B) \\ \sin A\cos B + \cos A\sin B & \cos A\cos B - \sin A\sin B \end{bmatrix}$$

Equating corresponding matrix elements from any column or row gives:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\text{and } \cos(A+B) = \cos A\cos B - \sin A\sin B$$

as required.

7. (a) The 180° rotation is represented by

$$\begin{bmatrix} \cos 180^\circ & -\sin 180^\circ \\ \sin 180^\circ & \cos 180^\circ \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

This transformation leaves point O unchanged at the origin. We need to transform this point to $(6,4)$ so the total transformation is represented by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

(b) Let $\theta$ be the angle that the line OO′ makes with the $x$-axis. This is the angle that O′A′B′C′ must be rotated clockwise in order to transform O′ onto the $x$-axis. Since O′ has coordinates $(6,4)$,

$$\tan\theta = \frac{4}{6} = \frac{2}{3}$$
$$\sin\theta = \frac{4}{\sqrt{4^2+6^2}} = \frac{4}{\sqrt{52}} = \frac{4}{2\sqrt{13}} = \frac{2}{\sqrt{13}}$$
$$\cos\theta = \frac{6}{2\sqrt{13}} = \frac{3}{\sqrt{13}}$$

The matrix to achieve this clockwise rotation (see question 3) is

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

(c) $\frac{1}{\sqrt{13}}\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 6 & 5 & 5 & 6 \\ 4 & 4 & 3 & 3 \end{bmatrix} = \frac{1}{\sqrt{13}}\begin{bmatrix} 26 & 23 & 21 & 24 \\ 0 & 2 & -1 & -3 \end{bmatrix}$

- O″$(\frac{26}{\sqrt{13}}, 0) = (2\sqrt{13}, 0)$,
- A″$(\frac{23}{\sqrt{13}}, \frac{2}{\sqrt{13}})$,
- B″$(\frac{21}{\sqrt{13}}, -\frac{1}{\sqrt{13}})$,
- C″$(\frac{24}{\sqrt{13}}, -\frac{3}{\sqrt{13}})$.

(You could, if preferred, give these with rational denominators and arrive at the same answers Sadler gives.)